Non-linear system of simultaneous equations

1. Solve for real numbers x, y :

$$\begin{cases} 4x^2 + y^2 = 17 + 4x \\ (2x-1)^2 + (y-8)^2 = 34 \end{cases}$$

$$\begin{cases} 4x^2 + y^2 = 17 + 4x \quad \ldots.(1) \\ (2x-1)^2 + (y-8)^2 = 34 \quad \ldots.(2) \end{cases}$$

From (2), $\quad 4x^2 - 4x + 1 + y^2 - 16y + 64 = 34$

$4x^2 + y^2 - 16y + 31 = 4x \quad \ldots.(3)$

$(1) - (3)$, $\quad 16y - 31 = 17$

$\therefore\ y = 3 \quad \ldots.(4)$

$(4) \downarrow (1)$, $\quad 4x^2 + 9 = 17 + 4x$

$4x^2 - 4x - 8 = 0$

$\therefore\ x = -1 \text{ or } x = 2$

The solution is $\quad (x, y) = (-1,3)$ or $(2,3)$

2. Solve for real numbers:

$$\begin{cases} w + x + y + z = 10 \\ w^2 + x^2 + y^2 + z^2 = 30 \\ w^3 + x^3 + y^3 + z^2 = 100 \\ wxyz = 24 \end{cases}$$

By inspection, $(w, x, y, z) = (1,2,3,4)$ is a solution of first and fourth equations. By substitution, it also satisfies the second and third equations. Since the equations are symmetric, all permutations are solutions, that is

$$(w, x, y, z) = (1,2,3,4) = (1,2,4,3) = (1,3,2,4) = \cdots (4,3,2,1)$$

However, these are all the solutions since the product of the degrees of the equations is $4! = 24$.

3. Given:

$$\begin{cases} a^2 + b^2 + ab = 9 \\ b^2 + c^2 + bc = 16 \\ c^2 + a^2 + ca = 25 \end{cases}$$

(a) If $\quad a, b, c > 0$, $\quad$ find $ab + bc + ca$.

(b) **(Hard)** If $\quad a, b, c$ are real numbers, $\quad$ find $ab + bc + ca$.

(a) Put

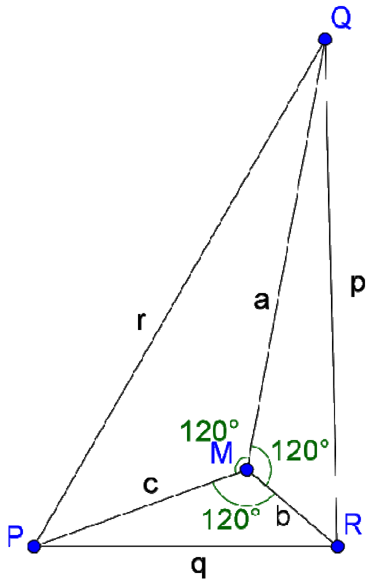
$$\begin{cases} p^2 = a^2 + b^2 + ab = a^2 + b^2 - 2ab\cos 120° = 9 \\ q^2 = b^2 + c^2 + bc = b^2 + c^2 - 2bc\cos 120° = 16 \\ r^2 = c^2 + a^2 + ca = c^2 + a^2 - 2ca\cos 120° = 25 \end{cases}$$

Since $p^2 + q^2 = 9 + 16 = 3^2 + 4^2 = 25 = 5^2 = r^2$, $p = 3, q = 4, r = 5$

By the converse of Pythagoras Theorem, we can form ΔPQR right-angled at R.

M is a point inside ΔPQR such that

$$\angle PMQ = \angle QMR = \angle RMP = 120°$$



Area of ΔPQR

$= \text{Area of } \Delta RMP + \text{Area of } \Delta QMR + \text{Area of } \Delta PMQ$

$$\frac{1}{2} \times 3 \times 4 = \frac{1}{2}ab\sin 120° + \frac{1}{2}bc\sin 120° + \frac{1}{2}ca\sin 120°$$

$$= \frac{1}{2}ab\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}bc\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}ca\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore\ ab + bc + ca = 6 \times \frac{4}{\sqrt{3}} = \underline{\underline{8\sqrt{3}}}$$

(b) (i) If $a, b, c < 0$, let $x = -a, y = -b, z = -c$ then the equation becomes:

$$\begin{cases} x^2 + y^2 - xy = 9 \\ y^2 + z^2 - yz = 16 \\ z^2 + x^2 - zx = 25 \end{cases}$$

which is similar to the original set of equation,

$$\therefore\ xy + yz + zx = ab + bc + ca = \underline{\underline{8\sqrt{3}}}$$

(ii) If $a, c > 0,\ b < 0$, let $x = a, y = -b, z = c$ then the equation becomes:

$$\begin{cases} x^2 + y^2 - xy = 9 \\ y^2 + z^2 - yz = 16 \\ z^2 + x^2 + zx = 25 \end{cases}$$

Put

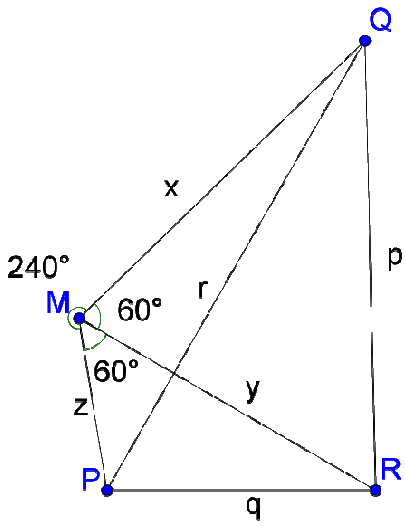
$$\begin{cases} p^2 = x^2 + y^2 - xy = x^2 + y^2 - 2xy\cos 60° = 9 \\ q^2 = y^2 + z^2 - yz = y^2 + z^2 - 2yz\cos 60° = 16 \\ r^2 = z^2 + x^2 + zx = z^2 + x^2 - 2zx\cos 120° = 25 \end{cases}$$

Since $p^2+q^2=9+16=3^2+4^2=25=5^2=r^2,\quad p=3, q=4, r=5$

By the converse of Pythagoras Theorem, we can form ΔPQR right-angled at R.

M is a point outside ΔPQR such that

$$\angle PMQ=120°, \angle QMR=\angle RMP=60°$$



Area of ΔPQR

$=$ Area of ΔRMP + Area of ΔQMR − Area of ΔPMQ

$$\frac{1}{2}\times 3\times 4=\frac{1}{2}xy\sin 60°+\frac{1}{2}yx\sin 60°-\frac{1}{2}zx\sin 120°$$

$$=\frac{1}{2}xy\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{2}yz\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}zx\left(\frac{\sqrt{3}}{2}\right)$$

$$=-\frac{1}{2}ab\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}bc\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}ca\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore\ ab+bc+ca=-6\times\frac{4}{\sqrt{3}}=\underline{\underline{-8\sqrt{3}\approx -13.856406460551}}$$

(iii) If $a<0,\ b,c>0$, let $x=a, y=-b, z=-c$

we can use similar method as in **(ii)**. (Readers may try.)

and can get $ab+bc+ca=\underline{\underline{-8\sqrt{3}\approx -13.856406460551}}$

(iv) We don't have solutions for other cases such as $a,b>0, c<0$.

Tough readers may also try to find the values for a, b, c.
I include here the complete solution.

$a\approx -1.00909,\quad b\approx 3.37444,\quad c\approx -4.4185,\quad d\approx -13.8564$

$a\approx 1.00909,\quad b\approx -3.37444,\quad c\approx 4.4185,\quad d\approx -13.8564$

$a\approx -2.354,\quad b\approx -1.02391,\quad c\approx -3.38852,\quad d\approx 13.8564$

$a\approx 2.354,\quad b\approx 1.02391,\quad c\approx 3.38852,\quad d\approx 13.8564$

where $d=ab+bc+ca$

4. **(Hard)** Solve for real numbers a, b, c :

$$\begin{cases} a^2 - 2ab + bc - c^2 + ca = 0 \\ b^2 - 2bc + ca - b^2 + ab = 0 \\ c^2 - 2ca + ab - a^2 + bc = 0 \end{cases}$$

The system is cyclic, without loss of generality, let $a \geq b \geq c$.
If $b = c$, then

$$\begin{cases} a^2 - 2ab + bc - c^2 + ca = 0 \\ b^2 - 2bc + ca - b^2 + ab = 0 \\ c^2 - 2ca + ab - a^2 + bc = 0 \end{cases} \Rightarrow \begin{cases} a^2 - 2ab + b^2 - b^2 + ab = 0 \\ b^2 - 2b^2 + ab - b^2 + ab = 0 \\ b^2 - 2ba + ab - a^2 + b^2 = 0 \end{cases} \Rightarrow \begin{cases} a^2 - ab = 0 \\ 2ab - 2b^2 = 0 \\ -a^2 - ab + 2b^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a(a-b) = 0 \\ 2b(a-b) = 0 \\ (2b+a)(b-a) = 0 \end{cases} \Rightarrow a = b \text{ , (a, b may be 0)}$$

Therefore $a = b = c$
Putting $a = b = c$ in the original system confirms the result.

Since the system is cyclic, if $a = b$, we can still give $a = b = c$ as solution.

Thus we can assume distinct values of a, b, c, that is, $a > b > c$.
Then we can replace $a = b + x, b = c + y, \quad x, y > 0 \quad \therefore a = c + x + y,$

$$\begin{cases} a^2 - 2ab + bc - c^2 + ca = 0 \\ b^2 - 2bc + ca - b^2 + ab = 0 \\ c^2 - 2ca + ab - a^2 + bc = 0 \end{cases}$$

$$\begin{cases} (c+x+y)^2 - 2(c+x+y)(c+y) + (c+y)c - c^2 + c(c+x+y) = 0 \\ (c+y)^2 - 2(c+y)c + c(c+x+y) - (c+y)^2 + (c+x+y)(c+y) = 0 \\ c^2 - 2c(c+x+y) + (c+x+y)(c+y) - (c+x+y)^2 + (c+y)c = 0 \end{cases}$$

$$\begin{cases} x^2 + cx - y^2 = 0 \\ xy + 2cx + y^2 + cy = 0 \\ -x^2 - xy - 3cx - cy = 0 \end{cases}$$

$$\begin{cases} c = -\dfrac{x^2 - y^2}{x}, \quad \text{where } x \neq 0 \ldots.(1) \\ c = -\dfrac{xy + y^2}{2x + y}, \text{where } y \neq -2x \ldots.(2) \\ c = -\dfrac{x^2 + xy}{3x + y}, \text{where } y \neq -3x \ldots.(3) \end{cases}$$

So if $x \neq 0,\ y \neq -2x,\ y \neq -3x$ (Note that $x, y > 0$, this holds.)

$(1)=(2)$, $\frac{x^2-y^2}{x}=\frac{x\,y+y^2}{2\,x+y}$ (When we put $(2)=(3)$, we get the same equation.)

$$(x^2-y^2)(2\,x+y)=x(x\,y+y^2)$$
$$(x+y)(x-y)(2\,x+y)=xy(x+y)$$
$$(x+y)(x-y)(2\,x+y)-xy(x+y)=0$$
$$(x+y)[(x-y)(2\,x+y)-xy]=0$$
$$(x+y)[2x^2-2xy-y^2]=0$$

(a) If $x+y=0$, then $x=-y$, but $x,y>0$. The solution is rejected.

(b) If $2x^2-2xy-y^2=0$ or $y^2+2xy-2x^2=0$

Using quadratic equation formula, $y=-x\pm\sqrt{3x^2}$

But since $x,y>0$, we have $y=\left(\sqrt{3}-1\right)x$ ....(4)

$(4)\downarrow(1)$, $c=-\frac{x^2-\left(\sqrt{3}-1\right)^2x^2}{x}=\left(3-2\sqrt{3}\right)x$

$$b=c+y=\left(3-2\sqrt{3}\right)x+\left(\sqrt{3}-1\right)x=\left(2-\sqrt{3}\right)x$$
$$a=b+x=\left(2-\sqrt{3}\right)x+x=\left(3-\sqrt{3}\right)x$$

$(a,b,c)=\left(\left(3-\sqrt{3}\right)x,\left(2-\sqrt{3}\right)x,\left(3-2\sqrt{3}\right)x\right)$, where x is a free parameter.

Complete solution:

$(a,b,c)=(\lambda,\lambda.\lambda)$ or $\left(\left(3-\sqrt{3}\right)\mu,\left(2-\sqrt{3}\right)\mu,\left(3-2\sqrt{3}\right)\mu\right)$, where λ,μ are parameters.

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